

NUMERICAL SOLUTIONS OF A BOUNDARY VALUE PROBLEM ON THE SPHERE USING RADIAL BASIS FUNCTIONS

Q. T. LE GIA [†]

Abstract. Boundary value problems on the unit sphere arise naturally in geophysics and oceanography when scientists model a physical quantity on large scales. Robust numerical methods play an important role in solving these problems. In this article, we construct numerical solutions to a boundary value problem defined on a spherical sub-domain (with a sufficiently smooth boundary) using radial basis functions (RBF). The error analysis between the exact solution and the approximation is provided. Numerical experiments are presented to confirm theoretical estimates.

1. Introduction. Boundary value problems on the unit sphere arise naturally in geophysics and oceanography when scientists model a physical quantity on large scales. In that situation, the curvature of the Earth cannot be ignored, and a boundary value problem has to be formulated on a subdomain of the unit sphere. For example, the study of planetary-scale oceanographic flows in which oceanic eddies interact with topography such as ridges and land masses or evolve in closed basin lead to the study of point vortices on the surface of the sphere with walls [14, 4]. Such vortex motions can be described as a Dirichlet problem on a subdomain of the sphere for the Laplace-Beltrami operator [5, 29]. Solving the problem exactly via conformal mapping methods onto the complex plane was proposed by Crowdy in [5]. Kidambi and Newton [29] also considered such a problem, assuming the sub-surface of the sphere lent itself to method of images. A boundary integral method for constructing numerical solutions to the problem was discussed in [13].

In this work, we propose a collocation method using spherical radial basis functions. Radial basis functions (RBFs) present a simple and effective way to construct approximate solutions to partial differential equations (PDEs) on spheres, via a collocation method [26] or a Galerkin method [22]. They have been used successfully for solving transport-like equations on the sphere [7, 8]. The method does not require a mesh, and is simple to implement.

While meshless methods using RBFs have been employed to derive numerical solutions for PDEs on the sphere only recently, it should be mentioned that approximation methods using RBFs for PDEs on bounded domains have been around for the last two decades. Originally proposed by Kansa [20, 21] for fluid dynamics, approximation methods for many types of PDEs defined on bounded domains in \mathbb{R}^n using RBFs have since been used widely [6, 10, 17, 18].

To the best of our knowledge, approximation methods using RBFs have not been investigated for boundary value problems defined on subdomains of the unit sphere. Given the potential of RBF methods on these problems, the present paper aims to present a collocation method for boundary value problems on the sphere and provide a mathematical foundation for error estimates.

The paper is organized as follows: in Section 2 we review some preliminaries on functions spaces, positive definite kernels, radial basis functions and the generalized interpolation problem on discrete point sets on the unit sphere. In Section 3 we define the boundary value problem on a spherical cap, then present a collocation method using spherical radial basis functions and our main result, Theorem 3.2. We conclude the paper by giving some numerical experiments in the last section.

[†]School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia qlegia@unsw.edu.au. The work of the author was supported by Australian Research Council.

Throughout the paper, we denote by c, c_1, c_2, \dots generic positive constants that may assume different values at different places, even within the same formula.

For two sequences $\{a_\ell\}_{\ell \in \mathbb{N}_0}$ and $\{b_\ell\}_{\ell \in \mathbb{N}_0}$, the notation $a_\ell \sim b_\ell$ means that there exist positive constants c_1 and c_2 such that $c_1 b_\ell \leq a_\ell \leq c_2 b_\ell$ for all $\ell \in \mathbb{N}_0$.

2. Preliminaries. Let \mathbb{S}^n be the *unit sphere*, i.e. $\mathbb{S}^n := \{\mathbf{x} \in \mathbb{R}^{n+1} : \|\mathbf{x}\| = 1\}$ in the Euclidean space \mathbb{R}^{n+1} , where $\|\mathbf{x}\| := \sqrt{\mathbf{x} \cdot \mathbf{x}}$ denotes the Euclidean norm of \mathbb{R}^{n+1} , induced by the Euclidean inner product $\mathbf{x} \cdot \mathbf{y}$ of two vectors \mathbf{x} and \mathbf{y} in \mathbb{R}^{n+1} . The surface area of the unit sphere \mathbb{S}^n is denoted by ω_n and is given by

$$\omega_n := |\mathbb{S}^n| = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}.$$

The *spherical distance* (or geodesic distance) $\text{dist}_{\mathbb{S}^n}(\mathbf{x}, \mathbf{y})$ of two points $\mathbf{x} \in \mathbb{S}^n$ and $\mathbf{y} \in \mathbb{S}^n$ is defined as the length of a shortest geodesic arc connecting the two points. The geodesic distance $\text{dist}_{\mathbb{S}^n}(\mathbf{x}, \mathbf{y})$ is the angle in $[0, \pi]$ between the points \mathbf{x} and \mathbf{y} , thus

$$\text{dist}_{\mathbb{S}^n}(\mathbf{x}, \mathbf{y}) := \arccos(\mathbf{x} \cdot \mathbf{y}).$$

Let Ω be a open simply connected subdomain of the sphere. For a point set $X := \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\} \subset \mathbb{S}^n$, the (*global*) *mesh norm* h_X is given by

$$h_X = h_{X, \mathbb{S}^n} := \sup_{\mathbf{x} \in \mathbb{S}^n} \inf_{\mathbf{x}_j \in X} \text{dist}_{\mathbb{S}^n}(\mathbf{x}, \mathbf{x}_j),$$

and the *local mesh norm* $h_{X, \Omega}$ with respect to the subdomain Ω is defined by

$$h_{X, \Omega} := \sup_{\mathbf{x} \in \Omega} \inf_{\mathbf{x}_j \in X \cap \Omega} \text{dist}_{\mathbb{S}^n}(\mathbf{x}, \mathbf{x}_j).$$

The mesh norm $h_{X_2, \partial\Omega}$ of $X_2 \subset \partial\Omega$ along the boundary $\partial\Omega$ is defined by

$$h_{X_2, \partial\Omega} := \sup_{\mathbf{x} \in \partial\Omega} \inf_{\mathbf{x}_j \in X_2} \text{dist}_{\partial\Omega}(\mathbf{x}, \mathbf{x}_j), \quad (2.1)$$

where $\text{dist}_{\mathbf{x} \in \partial\Omega}$ is here the geodesic distance along the boundary $\partial\Omega$.

2.1. Sobolev spaces on the sphere. Let Ω be \mathbb{S}^n or an open measurable subset of \mathbb{S}^n . Let $L_2(\Omega)$ denote the Hilbert space of (real-valued) square-integrable functions on Ω with the inner product

$$\langle f, g \rangle_{L_2(\Omega)} := \int_{\Omega} f(\mathbf{x})g(\mathbf{x}) d\omega_n(\mathbf{x})$$

and the induced norm $\|f\|_{L_2(\Omega)} := \langle f, f \rangle_{L_2(\Omega)}^{1/2}$. Here $d\omega_n$ is the Lebesgue surface area element of the sphere \mathbb{S}^n .

The space of continuous functions on the sphere \mathbb{S}^n and on the closed subdomain $\overline{\Omega}$ are denoted by $C(\Omega)$ and $C(\overline{\Omega})$ and are endowed with the supremum norms

$$\|f\|_{C(\mathbb{S}^n)} := \sup_{\mathbf{x} \in \mathbb{S}^n} |f(\mathbf{x})| \quad \text{and} \quad \|f\|_{C(\overline{\Omega})} := \sup_{\mathbf{x} \in \overline{\Omega}} |f(\mathbf{x})|,$$

respectively.

A *spherical harmonic* of degree $\ell \in \mathbb{N}_0$ (for the sphere \mathbb{S}^n) is the restriction of a homogeneous harmonic polynomial on \mathbb{R}^{n+1} of exact degree ℓ to the unit sphere

\mathbb{S}^n . The vector space of all spherical harmonics of degree ℓ (and the zero function) is denoted by $\mathbb{H}_\ell(\mathbb{S}^n)$ and has the dimension $Z(n, \ell) := \dim(\mathbb{H}_\ell(\mathbb{S}^n))$ given by

$$Z(n, 0) = 1 \quad \text{and} \quad Z(n, \ell) = \frac{(2\ell + n - 1)\Gamma(\ell + n - 1)}{\Gamma(\ell + 1)\Gamma(n)} \quad \text{for } \ell \in \mathbb{N}.$$

By $\{Y_{\ell,k} : k = 1, 2, \dots, Z(n, \ell)\}$, we will always denote an $L_2(\mathbb{S}^n)$ -orthonormal basis of $\mathbb{H}_\ell(\mathbb{S}^n)$ consisting of spherical harmonics of degree ℓ . Any two spherical harmonics of different degree are orthogonal to each other, and the union of all sets $\{Y_{\ell,k} : k = 1, 2, \dots, Z(n, \ell)\}$ constitutes a complete orthonormal system for $L_2(\mathbb{S}^n)$. Thus any function $f \in L_2(\mathbb{S}^n)$ can be represented in $L_2(\mathbb{S}^n)$ -sense by its *Fourier series* (or Laplace series)

$$f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{Z(n,\ell)} \widehat{f}_{\ell,k} Y_{\ell,k},$$

with the Fourier coefficients $\widehat{f}_{\ell,k}$ defined by

$$\widehat{f}_{\ell,k} := \int_{\mathbb{S}^n} f(\mathbf{x}) Y_{\ell,k}(\mathbf{x}) d\omega_n(\mathbf{x}).$$

The *space of spherical polynomials of degree $\leq K$* (that is, the set of the restrictions to \mathbb{S}^n of all polynomials on \mathbb{R}^{n+1} of degree $\leq K$) is denoted by $\mathbb{P}_K(\mathbb{S}^n)$. We have $\mathbb{P}_K(\mathbb{S}^n) = \bigoplus_{\ell=0}^K \mathbb{H}_\ell(\mathbb{S}^n)$ and $\dim(\mathbb{P}_K(\mathbb{S}^n)) = Z(n+1, K) \sim (K+1)^n$.

Any orthonormal basis $\{Y_{\ell,k} : k = 1, 2, \dots, Z(n, \ell)\}$ of $\mathbb{H}_\ell(\mathbb{S}^n)$ satisfies the *addition theorem* (see [27, p.10])

$$\sum_{k=0}^{Z(n,\ell)} Y_{\ell,k}(\mathbf{x}) Y_{\ell,k}(\mathbf{y}) = \frac{Z(n, \ell)}{\omega_n} P_\ell(n+1; \mathbf{x} \cdot \mathbf{y}), \quad (2.2)$$

where $P_\ell(n+1; \cdot)$ is the *normalized Legendre polynomial* of degree ℓ in \mathbb{R}^{n+1} . The normalized Legendre polynomials $\{P_\ell(n+1; \cdot)\}_{\ell \in \mathbb{N}_0}$, form a complete orthogonal system for the space $L_2([-1, 1]; (1-t^2)^{(n-2)/2})$ of functions on $[-1, 1]$ which are square-integrable with respect to the weight function $w(t) := (1-t^2)^{(n-2)/2}$. They satisfy $P_\ell(n+1; 1) = 1$ and

$$\int_{-1}^{+1} P_\ell(n+1; t) P_k(n+1; t) (1-t^2)^{(n-2)/2} dt = \frac{\omega_n}{\omega_{n-1} Z(n, \ell)} \delta_{\ell,k}, \quad (2.3)$$

where $\delta_{\ell,k}$ is the Kronecker delta (defined to be one if $\ell = k$ and zero otherwise).

The *Laplace-Beltrami operator* Δ^* (for the unit sphere \mathbb{S}^n) is the angular part of the Laplace operator $\Delta = \sum_{j=1}^{n+1} \partial^2 / \partial x_j^2$ for \mathbb{R}^{n+1} . Spherical harmonics of degree ℓ on \mathbb{S}^n are eigenfunctions of $-\Delta^*$, more precisely,

$$-\Delta^* Y_\ell = \lambda_\ell Y_\ell \quad \text{for all } Y_\ell \in \mathbb{H}_\ell(\mathbb{S}^n) \quad \text{with} \quad \lambda_\ell := \ell(\ell + n - 1).$$

For $s \in \mathbb{R}_0^+$, the *Sobolev space* $H^s(\mathbb{S}^n)$ is defined by (see [24, Chapter 1, Remark 7.6])

$$H^s(\mathbb{S}^n) := \left\{ f \in L_2(\mathbb{S}^n) : \sum_{\ell=0}^{\infty} (1 + \lambda_\ell)^s \sum_{k=1}^{Z(n,\ell)} |\widehat{f}_{\ell,k}|^2 < \infty \right\}.$$

The space $H^s(\mathbb{S}^n)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{H^s(\mathbb{S}^n)} := \sum_{\ell=0}^{\infty} (1 + \lambda_{\ell})^s \sum_{k=1}^{Z(n,\ell)} \widehat{f}_{\ell,k} \widehat{g}_{\ell,k}$$

and the induced norm

$$\|f\|_{H^s(\mathbb{S}^n)} := \langle f, f \rangle_{H^s(\mathbb{S}^n)}^{1/2} = \sum_{\ell=0}^{\infty} (1 + \lambda_{\ell})^s \sum_{k=1}^{Z(n,\ell)} |\widehat{f}_{\ell,k}|^2. \quad (2.4)$$

If $s > n/2$, then $H^s(\mathbb{S}^n)$ is embedded into $C(\mathbb{S}^n)$, and the Sobolev space $H^s(\mathbb{S}^n)$ is a *reproducing kernel Hilbert space*. This means that there exists a kernel $K_s : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{R}$, the so-called reproducing kernel, with the following properties: (i) $K_s(\mathbf{x}, \mathbf{y}) = K_s(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}^n$, (ii) $K_s(\cdot, \mathbf{y}) \in H^s(\mathbb{S}^n)$ for all (fixed) $\mathbf{y} \in \mathbb{S}^n$, and (iii) the reproducing property

$$\langle f, K_s(\cdot, \mathbf{y}) \rangle_{H^s(\mathbb{S}^n)} = f(\mathbf{y}) \quad \text{for all } f \in H^s(\mathbb{S}^n) \text{ and all } \mathbf{y} \in \mathbb{S}^n.$$

Sobolev spaces on \mathbb{S}^n can also be defined using local charts (see [24]). Here we use a specific atlas of charts, as in [19].

Let \mathbf{z} be a given point on \mathbb{S}^n , the spherical cap centered at \mathbf{z} of radius θ is defined by

$$G(\mathbf{z}, \theta) = \{\mathbf{y} \in \mathbb{S}^n : \cos^{-1}(\mathbf{z} \cdot \mathbf{y}) < \theta\}, \quad \theta \in (0, \pi),$$

where $\mathbf{z} \cdot \mathbf{y}$ denotes the Euclidean inner product of \mathbf{z} and \mathbf{y} in \mathbb{R}^{n+1} .

Let $\hat{\mathbf{n}}$ and $\hat{\mathbf{s}}$ denote the north and south poles of \mathbb{S}^n , respectively. Then a simple cover for the sphere is provided by

$$U_1 = G(\hat{\mathbf{n}}, \theta_0) \quad \text{and} \quad U_2 = G(\hat{\mathbf{s}}, \theta_0), \quad \text{where } \theta_0 \in (\pi/2, 2\pi/3). \quad (2.5)$$

The stereographic projection $\sigma_{\hat{\mathbf{n}}}$ of the punctured sphere $\mathbb{S}^n \setminus \{\hat{\mathbf{n}}\}$ onto \mathbb{R}^n is defined as a mapping that maps $\mathbf{x} \in \mathbb{S}^n \setminus \{\hat{\mathbf{n}}\}$ to the intersection of the equatorial hyperplane $\{\mathbf{z} = 0\}$ and the extended line that passes through \mathbf{x} and $\hat{\mathbf{n}}$. The stereographic projection $\sigma_{\hat{\mathbf{s}}}$ based on $\hat{\mathbf{s}}$ can be defined analogously. We set

$$\psi_1 = \frac{1}{\tan(\theta_0/2)} \sigma_{\hat{\mathbf{s}}} |_{U_1} \quad \text{and} \quad \psi_2 = \frac{1}{\tan(\theta_0/2)} \sigma_{\hat{\mathbf{n}}} |_{U_2}, \quad (2.6)$$

so that ψ_k , $k = 1, 2$, maps U_k onto $B(0, 1)$, the unit ball in \mathbb{R}^n . We conclude that $\mathcal{A} = \{U_k, \psi_k\}_{k=1}^2$ is a C^∞ atlas of covering coordinate charts for the sphere. It is known (see [34]) that the stereographic coordinate charts $\{\psi_k\}_{k=1}^2$ as defined in (2.6) map spherical caps to Euclidean balls, but in general concentric spherical caps are not mapped to concentric Euclidean balls. The projection ψ_k , for $k = 1, 2$, does not distort too much the geodesic distance between two points $\mathbf{x}, \mathbf{y} \in \mathbb{S}^n$, as shown in [23].

With the atlas so defined, we define the map π_k which takes a real-valued function g with compact support in U_k into a real-valued function on \mathbb{R}^n by

$$\pi_k(g)(x) = \begin{cases} g \circ \psi_k^{-1}(x), & \text{if } x \in B(0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\{\chi_k : \mathbb{S}^n \rightarrow \mathbb{R}\}_{k=1}^2$ be a partition of unity subordinated to the atlas, i.e., a pair of non-negative infinitely differentiable functions χ_k on \mathbb{S}^n with compact support in U_k ,

such that $\sum_k \chi_k = 1$. For any function $f : \mathbb{S}^n \rightarrow \mathbb{R}$, we can use the partition of unity to write

$$f = \sum_{k=1}^2 (\chi_k f), \text{ where } (\chi_k f)(\mathbf{x}) = \chi_k(\mathbf{x})f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^n.$$

The Sobolev space $H^s(\mathbb{S}^n)$ is defined to be the set

$$\{f \in L_2(\mathbb{S}^n) : \pi_k(\chi_k f) \in H^s(\mathbb{R}^n) \quad \text{for } k = 1, 2\},$$

which is equipped with the norm

$$\|f\|_{H^s(\mathbb{S}^n)} = \left(\sum_{k=1}^2 \|\pi_k(\chi_k f)\|_{H^s(\mathbb{R}^n)}^2 \right)^{1/2}. \quad (2.7)$$

This $H^s(\mathbb{S}^n)$ norm is equivalent to the $H^s(\mathbb{S}^n)$ norm given previously in (2.4)(see [24]).

Let $\Omega \subset \mathbb{S}^n$ be an open connected set with sufficiently smooth boundary. In order to define the Sobolev spaces on Ω , let $D_k = \psi_k(\Omega \cap U_k)$ for $k = 1, 2$. The local Sobolev space $H^s(\Omega)$ is defined to be the set

$$f \in L_2(\Omega) : \pi_k(\chi_k f)|_{D_k} \in H^s(D_k) \text{ for } k = 1, 2, D_k \neq \emptyset,$$

which is equipped with the norm

$$\|f\|_{H^s(\Omega)} = \left(\sum_{k=1}^2 \|\pi_k(\chi_k f)|_{D_k}\|_{H^s(D_k)}^2 \right)^{1/2} \quad (2.8)$$

where, if $\Omega = \emptyset$, then we adopt the convention that $\|\cdot\|_{H^s(D_k)} = 0$.

It should be noted that if $s = m$ which is a positive integer, we can define the local Sobolev norm via the following formula

$$\|f\|_{H^m(\Omega)} = \left(\sum_{k=0}^m \langle \nabla^k f, \nabla^k f \rangle_{L_2(\Omega)} \right)^{1/2}, \quad (2.9)$$

where ∇ is the surface gradient on the sphere.

We observe that ([19]) that there exists a positive constant $C_{\mathcal{A}}$, depending on \mathcal{A} and the partition of unity $\{\chi_1, \chi_2\}$, such that the geodesic distance of $\text{supp}\{\chi_k\}$ from the boundary of U_k is strictly greater than $C_{\mathcal{A}}$. A spherical cap $G(\mathbf{z}, \theta)$ with $\theta < C_{\mathcal{A}}/3$ will have its closure being a subset of at least one of the open subsets U_1 or U_2 , defined by (2.5).

We need the following important result proved in [2], which is a special case of an extension theorem for Besov spaces defined on minimally smooth domains in \mathbb{R}^n .

THEOREM 2.1. *Let D be a bounded connected set with sufficient smooth boundary. Then there is an extension operator $E_D : H^s(D) \rightarrow H^s(\mathbb{R}^n)$, for all $s \geq 0$, with*

1. $E_D f|_D = f$ for all $f \in H^s(D)$,
2. $\|E_D f\|_{H^s(\mathbb{R}^n)} \leq C_s \|f\|_{H^s(D)}$.

Now we state an extension theorem for a local domain on the sphere.

THEOREM 2.2 (Extension operator). *Suppose Ω is a open connected which is a subset of a spherical cap $G(\mathbf{z}, \theta)$ for some $\mathbf{z} \in \mathbb{S}^n$ and $\theta < C_{\mathcal{A}}/3$. Let $s > \frac{n}{2}$. There is an extension operator $E : H^s(\Omega) \rightarrow H^s(\mathbb{S}^n)$, for all $0 \leq \sigma \leq s$ with*

1. $Ef|_{\Omega} = f$ for all $f \in H^{\sigma}(\Omega)$,
2. $\|Ef\|_{H^{\sigma}(\mathbb{S}^n)} \leq C_{\sigma}\|f\|_{H^{\sigma}(\Omega)}$.

Proof. The case when $\Omega = G(\mathbf{z}, \theta)$ and σ being an integer is given in [19, Theorem 4.3]. We modify the proof there to get our results. We define a candidate extension operator $E : H^{\sigma}(\Omega) \rightarrow H^{\sigma}(\mathbb{S}^n)$ by

$$Ef(\mathbf{x}) = \sum_{k=1}^2 E_{D_k}((\chi_k f) \circ \psi_k^{-1}(\mathbf{x})|_{D_k})(\psi_k(\mathbf{x}))1_{U_k}(\mathbf{x}),$$

where E_{D_k} is the extension theorem for D_k from Theorem 2.1 and 1_{U_k} is the characteristic function for the set U_k . If $\mathbf{x} \in \Omega$, that is, $\psi(\mathbf{x}) \in D_k$ for $k = 1, 2$, then by part 1 of Theorem 2.1,

$$E_{D_k}((\chi_k f) \circ \psi_k^{-1}|_{D_k})(\psi_k(\mathbf{x})) = (\chi_k f) \circ \psi_k^{-1}(\psi_k(\mathbf{x})) = (\chi_k f)(\mathbf{x}),$$

hence $Ef(\mathbf{x}) = \sum_{k=1}^2 (\chi_k f)(\mathbf{x}) = f(\mathbf{x})$ as required. In order to prove part 2, by using (2.7)–(2.8) and the fact that $\pi_k(\chi_k)$ and $\psi_k \circ \psi_j^{-1}$ are $C_0^{\infty}(\mathbb{R}^d)$ functions, we have

$$\begin{aligned} \|Ef\|_{H^{\sigma}(\mathbb{S}^n)}^2 &= \sum_{j=1}^2 \|\pi_j(\chi_j Ef)\|_{H^{\sigma}(\mathbb{R}^n)}^2 = \sum_{j=1}^2 \|\pi_j(\chi_j)\pi_j(Ef)\|_{H^{\sigma}(\mathbb{R}^n)}^2 \\ &\leq C_{\chi} \sum_{j=1}^2 \|\pi_j(Ef)\|_{H^{\sigma}(\mathbb{R}^n)}^2 \\ &= \sum_{j=1}^2 \left\| \sum_{k=1}^2 E_{D_k}((\chi_k f) \circ \psi_k^{-1}|_{D_k})(\psi_k \circ \psi_j^{-1}) \right\|_{H^{\sigma}(\mathbb{R}^n)}^2 \\ &= 2C_{\chi} \sum_{j=1}^2 \sum_{k=1}^2 \|E_{D_k}((\chi_k f) \circ \psi_k^{-1}|_{D_k})(\psi_k \circ \psi_j^{-1})\|_{H^{\sigma}(\mathbb{R}^n)}^2 \\ &\leq 2C_{\chi} c_{\mathcal{A}} \sum_{k=1}^2 \|E_{D_k}((\chi_k f) \circ \psi_k^{-1}|_{D_k})\|_{H^{\sigma}(\mathbb{R}^n)}^2. \end{aligned}$$

The function $(\chi_k f) \circ \psi_k^{-1}|_{D_k}$ belongs to $H^{\sigma}(D_k)$, for $k = 1, 2$, hence from Theorem 2.1, there exists a constant c_{σ} , independent of f , such that

$$\|Ef\|_{H^{\sigma}(\mathbb{S}^n)}^2 \leq 2C_{\chi} c_{\mathcal{A}} c_{\sigma}^2 \sum_{k=1}^2 \|(\chi_k f) \circ \psi_k^{-1}|_{D_k}\|_{H^{\sigma}(D_k)}^2.$$

Taking the square roots,

$$\|Ef\|_{H^{\sigma}(\mathbb{S}^n)} \leq C_{\sigma}\|f\|_{H^{\sigma}(\Omega)}, \text{ where } C_{\sigma} = c_{\sigma}\sqrt{2C_{\chi}c_{\mathcal{A}}}.$$

□

THEOREM 2.3 (Trace theorem). *Let $\Omega \subset \mathbb{S}^n$ be a local region with a sufficient smooth boundary. Then, the restriction of $f \in H^s(\Omega)$ to $\partial\Omega$ is well defined, belongs to $H^{s-1/2}(\partial\Omega)$, and satisfies*

$$\|f\|_{H^{s-1/2}(\partial\Omega)} \leq C\|f\|_{H^s(\Omega)}.$$

Proof. Let $\partial D_k = \psi_k(\partial\Omega \cap U_k)$ for $k = 1, 2$. Then,

$$\|f\|_{H^{s-1/2}(\partial\Omega)}^2 = \sum_{k=1}^2 \|(\pi_k \chi_k f)|_{\partial D_k}\|_{H^{s-1/2}(\partial D_k)}^2.$$

Using the trace theorem for bounded domains in \mathbb{R}^n [37, Theorem 8.7], there are constants $c_k > 0$ for $k = 1, 2$ so that

$$\|(\pi_k \chi_k f)|_{\partial D_k}\|_{H^{s-1/2}(\partial D_k)} \leq c_k \|(\pi_k \chi_k f)|_{D_k}\|_{H^s(D_k)}.$$

Hence

$$\|f\|_{H^{s-1/2}(\partial\Omega)}^2 \leq \max\{c_1^2, c_2^2\} \sum_{k=1}^2 \|(\pi_k \chi_k f)|_{D_k}\|_{H^s(D_k)}^2 = \max\{c_1^2, c_2^2\} \|f\|_{H^s(\Omega)}^2.$$

□

2.2. Positive definite kernels on the sphere and native spaces. A continuous real-valued kernel $\phi : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{R}$ is called *positive definite* on \mathbb{S}^n if (i) $\phi(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}^n$ and (ii) for every finite set of distinct points $X = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\}$ on \mathbb{S}^n , the symmetric matrix $[\phi(\mathbf{x}_i, \mathbf{x}_j)]_{i,j=1,2,\dots,N}$ is positive definite.

A kernel $\phi : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{R}$ defined via $\phi(\mathbf{x}, \mathbf{y}) := \Phi(\mathbf{x} \cdot \mathbf{y})$, $\mathbf{x}, \mathbf{y} \in \mathbb{S}^n$, with a univariate function Φ , is called a *zonal* kernel.

Since the normalized Legendre polynomials $\{P_\ell(n+1; \cdot)\}_{\ell \in \mathbb{N}_0}$, form a complete orthogonal system for $L_2([-1, 1]; (1-t^2)^{(n-2)/2})$, any function $\Phi \in L_2([-1, 1]; (1-t^2)^{(n-2)/2})$ can be expanded into a *Legendre series* (see (2.3) for the normalization)

$$\Phi(t) = \frac{1}{\omega_n} \sum_{\ell=0}^{\infty} a_\ell Z(n, \ell) P_\ell(n+1; t), \quad (2.10)$$

with the Legendre coefficients

$$a_\ell := \omega_{n-1} \int_{-1}^{+1} \Phi(t) P_\ell(n+1; t) (1-t^2)^{(n-2)/2} dt.$$

Due to (2.10) and the addition theorem (2.2), a zonal kernel $\phi(\mathbf{x}, \mathbf{y}) := \Phi(\mathbf{x} \cdot \mathbf{y})$, $\mathbf{x}, \mathbf{y} \in \mathbb{S}^n$, where $\Phi \in L_2([-1, 1]; (1-t^2)^{(n-2)/2})$, has the expansion

$$\phi(\mathbf{x}, \mathbf{y}) = \frac{1}{\omega_n} \sum_{\ell=0}^{\infty} a_\ell Z(n, \ell) P_\ell(n+1; \mathbf{x} \cdot \mathbf{y}) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{Z(n, \ell)} a_\ell Y_{\ell, k}(\mathbf{x}) Y_{\ell, k}(\mathbf{y}). \quad (2.11)$$

In this paper we will only consider *positive definite zonal continuous kernels* ϕ of the form (2.11) for which

$$\sum_{\ell=0}^{\infty} |a_\ell| Z(n, \ell) < \infty. \quad (2.12)$$

This condition implies that the sums in (2.11) converge uniformly.

In [1], a complete characterization of positive definite kernels is established: a kernel ϕ of the form (2.11) satisfying the condition (2.12) is positive definite *if and*

only if $a_\ell \geq 0$ for all $\ell \in \mathbb{N}_0$ and $a_\ell > 0$ for infinitely many even values of ℓ and infinitely many odd values of ℓ (see also [33] and [38]).

With each positive definite zonal continuous kernel ϕ of the form (2.11) and satisfying the condition (2.12), we associate a *native space*: Consider the linear space

$$F_\phi := \left\{ \sum_{j=1}^N \alpha_j \phi(\cdot, \mathbf{x}_j) : \alpha_j \in \mathbb{R}, \mathbf{x}_j \in \mathbb{S}^n, j = 1, 2, \dots, N; N \in \mathbb{N} \right\},$$

endowed with the inner product

$$\left\langle \sum_{j=1}^N \alpha_j \phi(\cdot, \mathbf{x}_j), \sum_{i=1}^M \beta_i \phi(\cdot, \mathbf{y}_i) \right\rangle_\phi := \sum_{j=1}^N \sum_{i=1}^M \alpha_j \beta_i \phi(\mathbf{x}_j, \mathbf{y}_i)$$

and the associated norm $\|f\|_\phi := \langle f, f \rangle_\phi^{1/2}$. The *native space* \mathcal{N}_ϕ associated with ϕ is now defined as the completion of F_ϕ with respect to the norm $\|\cdot\|_\phi$. By construction, the native space \mathcal{N}_ϕ is a Hilbert space, and we will denote its inner product and norm also by $\langle \cdot, \cdot \rangle_\phi$ and $\|\cdot\|_\phi$, respectively.

The native space \mathcal{N}_ϕ is a (real) *reproducing kernel Hilbert space* with the reproducing kernel ϕ . This means that (i) ϕ is symmetric, (ii) $\phi(\cdot, \mathbf{y}) \in \mathcal{N}_\phi$ for all (fixed) $\mathbf{y} \in \mathbb{S}^n$, and (iii) the *reproducing property* holds, that is,

$$\langle f, \phi(\cdot, \mathbf{y}) \rangle_\phi = f(\mathbf{y}), \quad \text{for all } f \in \mathcal{N}_\phi \text{ and all } \mathbf{y} \in \mathbb{S}^n. \quad (2.13)$$

It is known that the native space \mathcal{N}_ϕ associated with a positive definite continuous zonal kernel ϕ , given by (2.11) and satisfying the conditions (2.12) and $a_\ell > 0$ for all $\ell \in \mathbb{N}_0$, can be described by

$$\mathcal{N}_\phi = \left\{ f \in L_2(\mathbb{S}^n) : \sum_{\ell=0}^{\infty} \sum_{k=1}^{Z(n,\ell)} \frac{|\widehat{f}_{\ell,k}|^2}{a_\ell} < \infty \right\},$$

equipped with the inner product

$$\langle f, g \rangle_\phi = \sum_{\ell=0}^{\infty} \sum_{k=1}^{Z(n,\ell)} \frac{\widehat{f}_{\ell,k} \widehat{g}_{\ell,k}}{a_\ell}$$

and the associated norm

$$\|f\|_\phi = \langle f, f \rangle_\phi^{1/2} = \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{Z(n,\ell)} \frac{|\widehat{f}_{\ell,k}|^2}{a_\ell} \right)^{1/2}. \quad (2.14)$$

If $a_\ell > 0$ for all $\ell \in \mathbb{N}_0$, we can conclude, from the assumption (2.12), that the Fourier series of any $f \in \mathcal{N}_\phi$ converges uniformly and that the native space \mathcal{N}_ϕ is embedded into $C(\mathbb{S}^n)$.

Comparing (2.14) with (2.4), we see that if $a_\ell \sim (1 + \lambda_\ell)^{-s}$, then $\|\cdot\|_\phi$ and $\|\cdot\|_{H^s(\mathbb{S}^n)}$ are equivalent norms, and hence \mathcal{N}_ϕ and $H^s(\mathbb{S}^n)$ are the same space.

2.3. Generalized interpolation with RBFs. Let $\phi : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{R}$ be a positive definite zonal continuous kernel given by (2.11) and satisfying the condition (2.12). Since the native space \mathcal{N}_ϕ is a reproducing kernel Hilbert space with reproducing kernel ϕ , any continuous linear functional \mathcal{L} on \mathcal{N}_ϕ has the representer $\mathcal{L}_2\phi(\cdot, \cdot)$. (Here the index 2 in $\mathcal{L}_2\phi(\cdot, \cdot)$ indicates that \mathcal{L} is applied to the kernel ϕ as a function of its second argument. Likewise $\mathcal{L}_1\phi(\cdot, \cdot)$ will indicate that \mathcal{L} is applied to the kernel ϕ as a function of its first argument.)

For a linearly independent set $\Xi = \{\mathcal{L}^1, \mathcal{L}^2, \dots, \mathcal{L}^N\}$ of continuous linear functionals on \mathcal{N}_ϕ , the *generalized radial basis function (RBF) interpolation problem* can be formulated as follows: Given the values $\mathcal{L}^1 f, \mathcal{L}^2 f, \dots, \mathcal{L}^N f$ of a function $f \in \mathcal{N}_\phi$, find the function $\Lambda_\Xi f$ in the N -dimensional approximation space

$$V_\Xi := \text{span} \left\{ \mathcal{L}_2^j \phi(\cdot, \cdot) : j = 1, 2, \dots, N \right\}$$

such that the conditions

$$\mathcal{L}^i(\Lambda_\Xi f) = \mathcal{L}^i f, \quad i = 1, 2, \dots, N, \quad (2.15)$$

are satisfied. We will call the function $\Lambda_\Xi f \in V_\Xi$ the *radial basis function approximant (RBF approximant)* of f .

Writing the RBF approximant $\Lambda_\Xi f$ as

$$\Lambda_\Xi f(\mathbf{x}) = \sum_{j=1}^N \alpha_j \mathcal{L}_2^j \phi(\mathbf{x}, \cdot), \quad \mathbf{x} \in \mathbb{S}^n,$$

the interpolation conditions (2.15) can therefore be written as

$$\sum_{j=1}^N \alpha_j \left\langle \mathcal{L}_2^j \phi(\cdot, \cdot), \mathcal{L}_2^i \phi(\cdot, \cdot) \right\rangle_\phi = \sum_{j=1}^N \alpha_j \mathcal{L}_1^i \mathcal{L}_2^j \phi(\cdot, \cdot) = \mathcal{L}^i f, \quad i = 1, 2, \dots, N. \quad (2.16)$$

Since $f \in \mathcal{N}_\phi$, we have $\mathcal{L}^i f = \langle f, \mathcal{L}_2^i \phi(\cdot, \cdot) \rangle_\phi$, $i = 1, 2, \dots, N$, and we see that $\Lambda_\Xi f$ is just the *orthogonal projection* of $f \in \mathcal{N}_\phi$ onto the approximation space V_Ξ with respect to $\langle \cdot, \cdot \rangle_\phi$. Therefore,

$$\|f - \Lambda_\Xi f\|_\phi \leq \|f\|_\phi. \quad (2.17)$$

The linear system has always a unique solution, because its matrix

$$[\mathcal{L}_1^i \mathcal{L}_2^j \phi(\cdot, \cdot)]_{i,j=1,2,\dots,N}$$

is the Gram matrix of the representers of the linearly independent functionals in Ξ .

We observe here that the linear system (2.16) can be solved for any given data set $\{\mathcal{L}^i f : i = 1, 2, \dots, N\}$, where the data does not necessarily has to come from a function in the native space \mathcal{N}_ϕ , but may come from any function f for which $\mathcal{L}^i f$ is well-defined for all $i = 1, 2, \dots, N$. Even if f is not in the native space we will use the notation $\Lambda_\Xi f$ for the solution of the generalized RBF interpolation problem (2.15).

2.4. Sobolev bounds for functions with scattered zeros. We need the following results from [15] concerning functions with scattered zeros on a subdomain of a Riemannian manifold.

THEOREM 2.4. *Let \mathbb{M} be a Riemannian manifold, $\Omega \subset \mathbb{M}$ be a bounded, Lipschitz domain that satisfies a certain uniform cone condition. Let X be a discrete set with sufficiently small mesh norm h . If $u \in W_p^m(\Omega)$ satisfies $u|_X = 0$, then we have*

$$\|u\|_{W_p^k(\Omega)} \leq C_{m,k,p,\mathbb{M}} h^{m-k} \|u\|_{W_p^m(\Omega)}$$

and

$$\|u\|_{L^\infty(\Omega)} \leq C_{m,k,p,\mathbb{M}} h^{m-d/p} \|u\|_{W_p^m(\Omega)}.$$

3. Boundary value problems on the sphere. After all these preparations we can formulate a boundary value problem for an elliptic differential operators L . Our standard application (and numerical example in Section 4) will be $L = \kappa^2 I - \Delta^*$, where I is the identity operator and κ is some fixed constant, on simply connected subregion Ω on \mathbb{S}^n with a Lipschitz boundary $\partial\Omega$. This partial differential equation occurs, for example, when solving the heat equation and the wave equation with separation of variables (for $\kappa \neq 0$) or in studying the vortex motion on the sphere (for $\kappa = 0$).

Let $s > 2$, and let Ω be a simply connected subregion with a Lipschitz boundary. Assume that the functions $f \in W_2^{s-2}(\Omega)$ and $g \in C(\partial\Omega)$ are given. We consider the following Dirichlet problem

$$Lu = f \text{ on } \Omega \quad \text{and } u = g \text{ on } \partial\Omega. \quad (3.1)$$

The existence and uniqueness of the solution to (3.1) follows from the general theory of existence and uniqueness of the solution to Dirichlet problems defined on Lipschitz domains in a Riemannian manifold [25].

LEMMA 3.1. *Let $n \geq 2$, and let Ω be a sub-domain on \mathbb{S}^n with a Lipschitz boundary. Let $L = \kappa^2 I - \Delta^*$ for some fixed constant $\kappa \geq 0$ and let $s \geq 2 + n/2$. Then L has the following properties:*

(i) *There exists a positive constant c such that*

$$\|Lf\|_{H^{s-2}(\Omega)} \leq c\|f\|_{H^s(\Omega)}.$$

(ii) *There exists a positive constant c such that*

$$\langle Lf, f \rangle_{L_2(\Omega)} \geq c\|f\|_{L_2(\Omega)}^2$$

for all $f \in W_2^s(\Omega) \cap C(\overline{\Omega})$ with $f = 0$ on $\partial\Omega$.

(iii) *There exists a positive constant c such that*

$$\|f\|_{C(\overline{\Omega})} \leq c\|f\|_{C(\partial\Omega)}$$

for all $f \in W_2^s(\Omega) \cap C(\overline{\Omega})$ which satisfy $Lf = 0$ on Ω .

Proof.

(i)

Suppose $s = m$, where m is an integer. Using definition (2.9) and the fact that $\Delta^* = -\nabla^* \nabla$, where ∇^* denote the surface divergent on the sphere, we have

$$\|Lu\|_{W_2^{s-2}(\Omega)}^2 = \sum_{k=0}^m \langle \nabla^k Lu, \nabla^k Lu \rangle_{L_2(\Omega)}$$

$$\begin{aligned}
&= \sum_{k=0}^{m-2} \langle \nabla^k(\kappa^2 u - \Delta^* u), \nabla^k(\kappa^2 u - \Delta^* u) \rangle_{L_2(\Omega)} \\
&= \sum_{k=0}^{m-2} \kappa^4 \langle \nabla^k u, \nabla^k u \rangle_{L_2(\Omega)} - 2\kappa^2 \langle \nabla^{k+1} u, \nabla^{k+1} u \rangle_{L_2(\Omega)} \\
&\quad + \langle \nabla^{k+2} u, \nabla^{k+2} u \rangle_{L_2(\Omega)} \\
&\leq \max\{\kappa^4, 2\kappa^2, 1\} \sum_{k=0}^m \langle \nabla^k u, \nabla^k u \rangle_{L_2(\Omega)} \\
&\leq C \|u\|_{W_2^s(\Omega)}^2.
\end{aligned}$$

The case that s is a real number follows from interpolation between bounded operators.

(ii) With the assumption on s , the Sobolev imbedding theorem for functions defined on Riemannian manifolds [16, p.34] implies that $W_2^s(\Omega) \subset C^2(\Omega)$.

From Green's first surface identity [12, (1.2.49)], or more generally, the first Green's formula for compact, connected, and oriented manifolds in \mathbb{R}^{n+1} [3, p.84], we find for any $f \in W_2^s(\Omega) \cap C(\overline{\Omega})$ with $f = 0$ on $\partial\Omega$ that

$$\begin{aligned}
\langle (\kappa^2 - \Delta^*)f, f \rangle_{L_2(\Omega)} &= \kappa^2 \|f\|_{L_2(\Omega)}^2 - \langle \Delta^* f, f \rangle_{L_2(\Omega)} \\
&= \kappa^2 \|f\|_{L_2(\Omega)}^2 + \|\nabla f\|_{L_2(\Omega)}^2 - \int_{\partial\Omega} f(\mathbf{x}) \frac{\partial f(\mathbf{x})}{\partial \nu} d\sigma(\mathbf{x}) \\
&= \kappa^2 \|f\|_{L_2(\Omega)}^2 + \|\nabla f\|_{L_2(\Omega)}^2,
\end{aligned}$$

where ∇ is the surface gradient, ν the (external) unit normal on the boundary $\partial\Omega$, and $d\sigma$ the curve element of the boundary (curve) $\partial\Omega$. From the Poincaré inequality for a bounded domain on a Riemannian manifold [32],

$$\|\nabla f\|_{L_2(\Omega)} \geq c \|f\|_{L_2(\Omega)}$$

for all $f \in W_2^s(\Omega) \cap C(\overline{\Omega})$ with $f = 0$ on $\partial\Omega$. Thus

$$\langle (\kappa^2 - \Delta^*)f, f \rangle_{L_2(\Omega)} \geq (c + \kappa^2) \|f\|_{L_2(\Omega)}^2,$$

from which property (ii) is proved.

(iii)

The property (iii) follows from the maximum principle for elliptic PDEs on manifolds. From [30, Theorem 9.3], we know that every $g \in C^1(\Omega)$ which satisfies

$$\Delta^* g - \kappa^2 g \leq 0 \quad \text{on } \Omega \quad \text{and} \quad g \geq 0 \quad \text{on } \Omega$$

in distributional sense satisfies the *strong maximum principle*, that is, if $g(\mathbf{y}_0) = 0$ for some $\mathbf{y}_0 \in \Omega$ then $g \equiv 0$ in Ω . In particular, this implies if $g \in C^1(\Omega) \cap C(\overline{\Omega})$ that g assumes its zeros on the boundary.

In our case $f \in W_2^s(\Omega) \cap C(\overline{\Omega})$, and since $W_2^s(\Omega) \subset C^2(\Omega)$, we consider (twice differentiable) classical solutions of $\kappa^2 f - \Delta^* f = 0$. From the strong maximum principle we may conclude that every $f \in W_2^s(\Omega) \cap C(\overline{\Omega})$ that satisfies $\kappa^2 f - \Delta^* f = 0$ has the property

$$\sup_{\mathbf{x} \in \overline{\Omega}} |f(\mathbf{x})| = \sup_{\mathbf{x} \in \partial\Omega} |f(\mathbf{x})|, \quad (3.2)$$

which establishes property (iii) in the Theorem

This can be seen as follows: Consider $f \in W_2^s(\Omega) \cap C(\overline{\Omega})$ that satisfies $\kappa^2 f - \Delta^* f = 0$. Let $\mathbf{y}_1 \in \overline{\Omega}$ and $\mathbf{y}_2 \in \overline{\Omega}$ be such that

$$f(\mathbf{y}_1) = \min_{\mathbf{y} \in \overline{\Omega}} f(\mathbf{y}) \leq f(\mathbf{x}) \leq \max_{\mathbf{y} \in \overline{\Omega}} f(\mathbf{y}) = f(\mathbf{y}_2) \quad \text{for all } \mathbf{x} \in \overline{\Omega}.$$

Then

$$\sup_{\mathbf{x} \in \overline{\Omega}} |f(\mathbf{x})| = \begin{cases} f(\mathbf{y}_2) & \text{if } f \geq 0 \text{ on } \overline{\Omega}, \\ -f(\mathbf{y}_1) & \text{if } f \leq 0 \text{ on } \overline{\Omega}, \\ \max\{-f(\mathbf{y}_1), f(\mathbf{y}_2)\} & \text{if } f \text{ assumes negative and positive values.} \end{cases} \quad (3.3)$$

If $f(\mathbf{y}_1) \leq 0$, consider $g_1(\mathbf{x}) := f(\mathbf{x}) - f(\mathbf{y}_1)$. Then $g_1(\mathbf{y}_1) = 0$ and $g_1(\mathbf{x}) \geq 0$ on $\overline{\Omega}$, and we have

$$(\Delta^* - \kappa^2)g_1 = (\Delta^* - \kappa^2)f + \kappa^2 f(\mathbf{y}_1) = \kappa^2 f(\mathbf{y}_1) \leq 0.$$

Thus the strong maximum principle implies that g_1 assumes its zeros on the boundary and hence $\mathbf{y}_1 \in \partial\Omega$. If $f(\mathbf{y}_2) \geq 0$, consider $g_2(\mathbf{x}) := f(\mathbf{x}) - f(\mathbf{y}_2)$. Then $g_2(\mathbf{y}_2) = 0$ and $g_2(\mathbf{x}) \leq 0$ on $\overline{\Omega}$, and we find

$$(\Delta^* - \kappa^2)g_2 = -\kappa^2 f(\mathbf{y}_2) - (\Delta^* - \kappa^2)f = -\kappa^2 f(\mathbf{y}_2) \leq 0.$$

Thus the strong maximum principle implies that g_2 assumes its zeros on the boundary and hence $\mathbf{y}_2 \in \partial\Omega$. Thus (3.3) implies (3.2). \square

We now discuss a method to construct an approximate solution to the Dirichlet problem 3.1 using radial basis functions. Assume that the values of the functions f and g are given on the discrete sets $X_1 := \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_M\} \subset \Omega$ and $X_2 := \{\mathbf{x}_{M+1}, \dots, \mathbf{x}_N\} \subset \partial\Omega$, respectively. Furthermore, assume that the local mesh norm $h_{X_1, \Omega}$ of X_1 and the mesh norm $h_{X_2, \partial\Omega}$ of X_2 along the boundary $\partial\Omega$ (see (2.1) below) are sufficiently small. We wish to find an approximation of the solution $u \in W_2^s(\Omega) \cap C(\overline{\Omega})$ of the *Dirichlet boundary value problem*

$$Lu = f \quad \text{on } \Omega \quad \text{and} \quad u = g \quad \text{on } \partial\Omega.$$

Let $\Xi = \Xi_1 \cup \Xi_2$ with $\Xi_1 := \{\delta_{\mathbf{x}_j} \circ L : j = 1, 2, \dots, M\}$ and $\Xi_2 := \{\delta_{\mathbf{x}_j} : j = M+1, \dots, N\}$.

We choose a RBF ϕ such that $\mathcal{N}_\phi = H^s(\mathbb{S}^n)$ for some $s > 2 + \lfloor n/2 + 1 \rfloor$. Under the assumption that Ξ is a set of linearly independent functionals, we compute the RBF approximant $\Lambda_\Xi u$, defined by

$$\Lambda_\Xi u = \sum_{j=1}^M \alpha_j L_2 \phi(\cdot, \mathbf{x}_j) + \sum_{j=M+1}^N \alpha_j \phi(\cdot, \mathbf{x}_j), \quad (3.4)$$

in which the coefficients α_j , for $j = 1, \dots, N$, are computed from the *collocation conditions*

$$L(\Lambda_\Xi u)(\mathbf{x}_j) = f(\mathbf{x}_j), \quad j = 1, 2, \dots, M, \quad (3.5)$$

$$\Lambda_\Xi u(\mathbf{x}_j) = g(\mathbf{x}_j), \quad j = M+1, \dots, N. \quad (3.6)$$

We want to derive $L_2(\Omega)$ -error estimates between the approximation and the exact solution, which is stated in the following theorem.

THEOREM 3.2. Let $L = \kappa^2 I - \Delta^*$ for some fixed constant $\kappa \geq 0$ and let $s \geq 2 + \lfloor n/2 + 1 \rfloor$. Consider the Dirichlet boundary value problem

$$Lu = f \quad \text{on } \Omega \quad \text{and} \quad u = g \quad \text{on } \partial\Omega,$$

where we assume that the unknown solution u is in $W_2^s(\Omega) \cap C(\overline{\Omega})$ and that $f \in W_2^{s-2}(\Omega)$ and $g \in C(\partial\Omega)$. Assume that f is given on the point set $X_1 = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_M\} \subset \Omega$ with sufficiently small local mesh norm $h_{X_1, \Omega}$, and suppose that g is given on the point set $X_2 = \{\mathbf{x}_{M+1}, \dots, \mathbf{x}_N\} \subset \partial\Omega$ with sufficiently small mesh norm $h_{X_2, \partial\Omega}$. Let ϕ be a positive definite zonal continuous kernel of the form (2.11) for which

$$a_\ell \sim (1 + \lambda_\ell)^{-s}. \quad (3.7)$$

Let $\Lambda_\Xi u$ denote the RBF approximant (3.4) which satisfies the collocation conditions (3.5) and (3.6). Then

$$\|u - \Lambda_\Xi u\|_{L_2(\Omega)} \leq c \max\{h_{X_1, \Omega}^{s-2}, h_{X_2, \partial\Omega}^{s-n/2}\} \|u\|_{W_2^s(\Omega)}. \quad (3.8)$$

Our general approach follows the one discussed in [10], [11], and in [36, Chapter 16] for the case of boundary problems on subsets of \mathbb{R}^n . In contrast to the approach in [36, Chapter 16], where the error analysis is based on the power function, we also use the results on functions with scattered zeros (see Theorem 2.4) locally via the charts.

Proof.

Step 1. First we prove the following inequality using the ideas from [10, Theorem 5.1].

$$\|u - \Lambda_\Xi u\|_{L_2(\Omega)} \leq \|Lu - L(\Lambda_\Xi u)\|_{L_2(\Omega)} + c\|u - \Lambda_\Xi u\|_{C(\partial\Omega)}. \quad (3.9)$$

Since the boundary value problem has a unique solution, there exists a function $w \in W_2^s(\Omega) \cap C(\overline{\Omega})$ such that

$$Lw = Lu \quad \text{on } \Omega \quad \text{and} \quad w = \Lambda_\Xi u \quad \text{on } \partial\Omega. \quad (3.10)$$

From the triangle inequality,

$$\|u - \Lambda_\Xi u\|_{L_2(\Omega)} \leq \|u - w\|_{L_2(\Omega)} + \|w - \Lambda_\Xi u\|_{L_2(\Omega)} \quad (3.11)$$

Since $L(u - w) = 0$ on Ω (from (3.10)), the property (iii) and (3.10) imply

$$\|u - w\|_{L_2(\Omega)} \leq c\|u - w\|_{C(\overline{\Omega})} \leq c\|u - w\|_{C(\partial\Omega)} = c\|u - \Lambda_\Xi u\|_{C(\partial\Omega)}. \quad (3.12)$$

Since $w - \Lambda_\Xi u = 0$ on $\partial\Omega$ (from (3.10)), the property (ii) and the Cauchy-Schwarz inequality yield that

$$\begin{aligned} \|w - \Lambda_\Xi u\|_{L_2(\Omega)}^2 &\leq \langle L(w - \Lambda_\Xi u), w - \Lambda_\Xi u \rangle_{L_2(\Omega)} \\ &\leq \|L(w - \Lambda_\Xi u)\|_{L_2(\Omega)} \|w - \Lambda_\Xi u\|_{L_2(\Omega)}, \end{aligned}$$

thus implying

$$\|w - \Lambda_\Xi u\|_{L_2(\Omega)} \leq \|Lw - L(\Lambda_\Xi u)\|_{L_2(\Omega)} = \|Lu - L(\Lambda_\Xi u)\|_{L_2(\Omega)}, \quad (3.13)$$

where we have used $Lw = Lu$ on Ω in the last step. Applying (3.12) and (3.13) in (3.11) gives

$$\|u - \Lambda_{\Xi}u\|_{L_2(\Omega)} \leq c\|u - \Lambda_{\Xi}u\|_{C(\partial\Omega)} + \|Lu - L(\Lambda_{\Xi}u)\|_{L_2(\Omega)}$$

which proves (3.9).

Step 2. In this step, we will estimate the first term in the right hand side of (3.9). By using Theorem 2.4, we obtain

$$\begin{aligned} \|Lu - L(\Lambda_{\Xi}u)\|_{L_2(\Omega)} &\leq ch_{X_1, \Omega}^{s-2} \|Lu - L(\Lambda_{\Xi}u)\|_{W_2^{s-2}(\Omega)} \\ &\leq ch_{X_1, \Omega}^{s-2} \|u - \Lambda_{\Xi}u\|_{W_2^s(\Omega)}, \end{aligned} \quad (3.14)$$

where we have used the fact that $\|Lg\|_{W_2^{s-2}(\Omega)} \leq C\|g\|_{W^s(\Omega)}$, see Lemma 3.1 part i).

Next, our assumptions on the region Ω allow us to extend the function $u \in W_2^s(\Omega)$ to a function $Eu \in W_2^s(\mathbb{S}^n)$. Moreover, since $X \subset \Omega$ and $Eu|_{\Omega} = u|_{\Omega}$, the generalized interpolant $\Lambda_{\Xi}u$ coincides with the generalized interpolant $\Lambda_{\Xi}(Eu)$ on Ω . Finally, the Sobolev space norm on $W_2^s(\mathbb{S}^n)$ is equivalent to the norm induced by the kernel ϕ and the generalized interpolant is norm-minimal. This all gives

$$\begin{aligned} \|u - \Lambda_{\Xi}u\|_{W_2^s(\Omega)} &= \|Eu - \Lambda_{\Xi}Eu\|_{W_2^s(\Omega)} \leq \|Eu - \Lambda_{\Xi}Eu\|_{W_2^s(\mathbb{S}^n)} \\ &\leq \|Eu\|_{W_2^s(\mathbb{S}^n)} \leq C\|u\|_{W_2^s(\mathbb{S}^n)}, \end{aligned} \quad (3.15)$$

which establishes the stated interior error estimate.

Step 3. In this step, we will estimate the second term in the right hand side of (3.9). For the boundary estimate, by using Theorem 2.4 for $\partial\Omega$, which is manifold of dimension $n - 1$, we obtain

$$\|u - \Lambda_{\Xi}u\|_{C(\partial\Omega)} \leq ch_{X_2, \partial\Omega}^{s-1/2-(n-1)/2} \|u - \Lambda_{\Xi}u\|_{W_2^{s-1/2}(\partial\Omega)} \quad (3.16)$$

Using the trace theorem (Theorem 2.3) and (3.15), we have

$$\|u - \Lambda_{\Xi}u\|_{W_2^{s-1/2}(\partial\Omega)} \leq C\|u - \Lambda_{\Xi}u\|_{W_2^s(\Omega)} \leq C\|u\|_{W_2^s(\Omega)} \quad (3.17)$$

The boundary estimate then follows from (3.16)–(3.17).

The desired estimate will follow from results of all three steps. \square

4. Numerical experiments. In this section, we consider the following boundary value problem on the spherical cap of radius $\pi/3$ centered at the north pole:

$$\begin{aligned} Lu(\mathbf{x}) &:= -\Delta^*u(\mathbf{x}) + u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in S(\mathbf{n}; \pi/3), \\ u(\mathbf{x}) &= g(\mathbf{x}) & \mathbf{x} \in \partial S(\mathbf{n}; \pi/3). \end{aligned}$$

Let f be defined so that the exact solution is given by the Franke function [9] defined on the unit sphere \mathbb{S}^2 . To be more precise, let

$$\mathbf{x} = (x, y, z) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad \text{for} \quad \theta \in [0, \pi], \phi \in [0, 2\pi).$$

Then we define

$$u(\mathbf{x}) = 0.75 \exp\left(-\frac{(9x-2)^2 + (9y-2)^2}{4}\right) + 0.75 \exp\left(-\frac{(9x+1)^2}{49} - \frac{9y+1}{10}\right)$$

$$+ 0.5 \exp \left(-\frac{(9x-7)^2 + (9y-3)^2}{4} \right) - 0.2 \exp \left(-(9x-4)^2 - (9y-7)^2 \right)$$

and compute the function f via the formula

$$f(\mathbf{x}(\theta, \phi)) = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} + u(\mathbf{x}(\theta, \phi)).$$

A plot of the exact solution u is given in Figure 4.1. Even though the algorithm allows

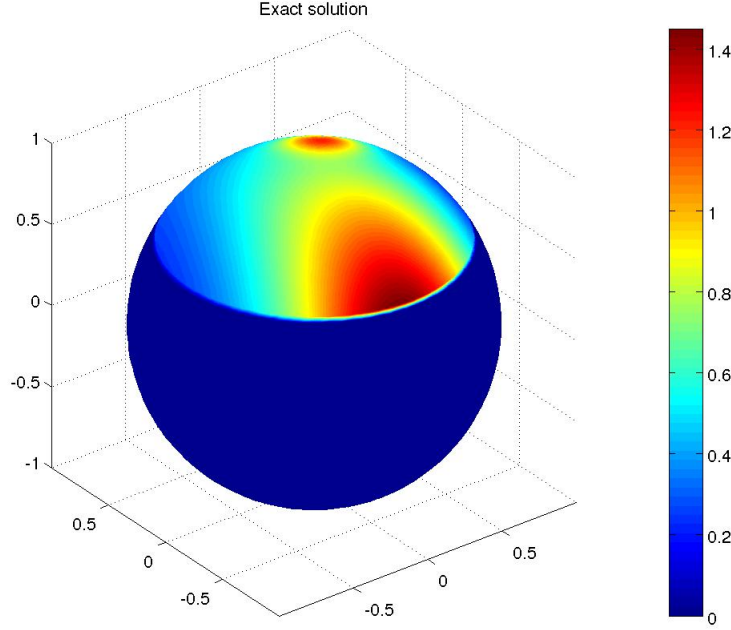


FIG. 4.1. *Exact solution*

the collocation points to be scattered freely on the sphere, choosing sets of collocation points distributed roughly uniformly over the whole sphere significantly improves the quality of the approximate solutions and condition numbers. To this end, the sets of points used to construct the approximate solutions are generated using the equal area partitioning algorithm [31].

The RBF used is

$$\psi(r) = (1-r)_+^8 (1 + 8r + 25r^2 + 32r^3)$$

and

$$\phi(\mathbf{x}, \mathbf{y}) = \psi(|\mathbf{x} - \mathbf{y}|) = \psi(\sqrt{2 - 2\mathbf{x} \cdot \mathbf{y}}).$$

It can be shown that ϕ is a kernel which satisfies condition (3.7) with $s = 9/2$ ([28]).

The kernel ϕ is a zonal function, i.e. $\phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} \cdot \mathbf{y})$ where $\Phi(t)$ is a univariate function. For zonal functions, the Laplace-Beltrami operator can be computed via

$$\Delta^* \Phi(\mathbf{x} \cdot \mathbf{y}) = \mathcal{L} \Phi(t), \quad t = \mathbf{x} \cdot \mathbf{y},$$

M	h_{X_1}	$\ e\ $	EOC
500	0.0733	2.9000E-03	
1000	0.0520	5.1602E-04	5.03
2000	0.0366	8.6364E-05	5.09
4000	0.0258	1.4596E-05	5.08

TABLE 4.1

Interior errors with a fixed number of boundary points $N - M = 200$

where

$$\mathcal{L} = \frac{d}{dt}(1 - t^2) \frac{d}{dt}$$

In our case,

$$\mathcal{L}\Phi(t) = -44(\sqrt{2-2t} - 1)^6 \left(6t^2 - 18t - t + 12 + \frac{208t^3 - 260t^2 - 92t + 144}{\sqrt{2-2t}} \right).$$

The normalized interior L_2 error $\|e\|$ is approximated by an ℓ_2 error, thus in principle we define (note that the area of the cap $\mathbb{S}^2(\mathbf{n}; \pi/3)$ is π)

$$\begin{aligned} \|e\| &:= \left(\frac{1}{\pi} \int_{\mathbb{S}^2(\mathbf{n}; \pi/3)} |u(\mathbf{x}) - \Lambda_{\Xi} u(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2} \\ &= \left(\frac{1}{\pi} \int_0^{\pi/3} \int_0^{2\pi} |u(\theta, \phi) - \Lambda_{\Xi} u(\theta, \phi)|^2 \sin \theta d\phi d\theta \right)^{1/2}, \end{aligned}$$

and in practice approximate this by the midpoint rule,

$$\left(\frac{1}{\pi} \frac{2\pi^2}{3|\mathcal{G}|} \sum_{\mathbf{x}(\theta, \phi) \in \mathcal{G}} |u(\theta, \phi) - \Lambda_{\Xi} u(\theta, \phi)|^2 \sin \theta \right)^{1/2},$$

where \mathcal{G} is a longitude-latitude grid in the interior of $\mathbb{S}^2(\mathbf{n}; \pi/3)$ containing the centers of rectangles of size 0.9 degree times 1.8 degree and $|\mathcal{G}| = 67 \times 200 = 13400$.

The supremum error $L^\infty(\partial S(\mathbf{n}; \pi/3))$ is approximated by

$$\|e\|_\infty = \max_{\mathbf{x} \in \mathcal{G}'} |u(\mathbf{x}) - \Lambda_{\Xi} u(\mathbf{x})|$$

in which \mathcal{G}' is a set of 3000 equally spaced points on $\partial S(\mathbf{n}; \pi/3)$.

As can be seen from in Tables 4.1 and 4.2, the numerical results show a better convergence rate predicted by Theorem 3.2.

Acknowledgments. The author is grateful to many helpful discussions with Dr. Kerstin Hesse when writing the earlier version of the paper. He would also like to thank Professor Francis Narcowich for pointing out the recent results on Sobolev bounds for functions with scattered zeros on a Riemannian manifold.

REFERENCES

- [1] D. Chen, V. A. Menegatto, X. Sun, A necessary and sufficient condition for strictly positive definite functions on spheres, Proc. Amer. Math. Society, **131** (2003), 2733–2740.

$N - M$	h_{X_2}	$\ e\ _\infty$	EOC
100	0.0272	2.7561E-05	
200	0.0136	7.0789E-08	8.60
400	0.0068	1.0812E-10	9.35
800	0.0034	8.4499E-13	7.00

TABLE 4.2

Boundary errors with a fixed number of interior points $M = 1000$

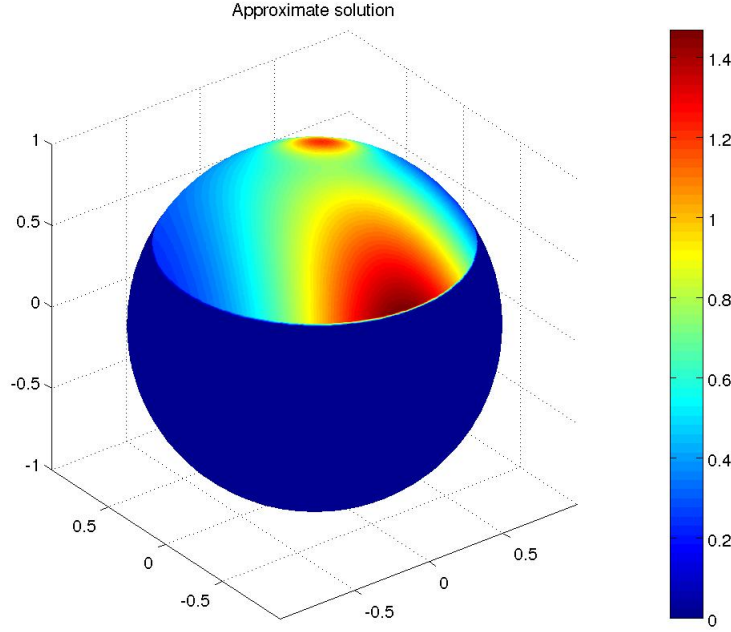


FIG. 4.2. Approximate solution with $M = 4000$ and $N = 4200$

- [2] R. A. DeVore, R. C. Sharpley, Besov spaces on domains in \mathbb{R}^d , Trans. Amer. Math. Soc., **335**(2) (1993), 843–864.
- [3] I. Agricola, T. Friedrich, Global Analysis: Differential forms in Analysis, Geometry and Physics, Graduate Studies in Mathematics, Vol. 52, Amer. Math. Society, Providence, Rhode Island.
- [4] Chaos, special focus issue “Large long-lived coherent structures out of chaos in planetary atmospheres and oceans”, Chaos **4** (1994).
- [5] D. Crowdy, Point vortex motion on the surface of a sphere with impenetrable boundaries, Physics of Fluids, **18**, (2006), 036602.
- [6] G. E. Fasshauer, Solving differential equations with radial basis functions: multilevel methods and smoothing. Advances in Comput. Math., **11**, (1999), 139–159.
- [7] N. Flyer and G. Wright, Transport schemes on a sphere using radial basis functions, J. Comp. Phys., **226** (2007), 1059–1084.
- [8] N. Flyer and G. Wright. A radial basis function method for the shallow water equations on a sphere, Proc. R. Soc. A, **465** (2009), 1949–1976.
- [9] R. Franke, A critical comparison of some methods for interpolation of scattered data, Technical Report NPS-53-79-003, Naval Postgraduate School, 1979.
- [10] C. Franke, R. Schaback: Solving partial differential equations by collocation using radial basis functions, Applied Math. Comput., **93** (1998), 73–82.
- [11] C. Franke, R. Schaback: Convergence order estimates of meshless collocation methods using radial basis functions, Adv. Comput. Math., **8** (1998), 381–399.

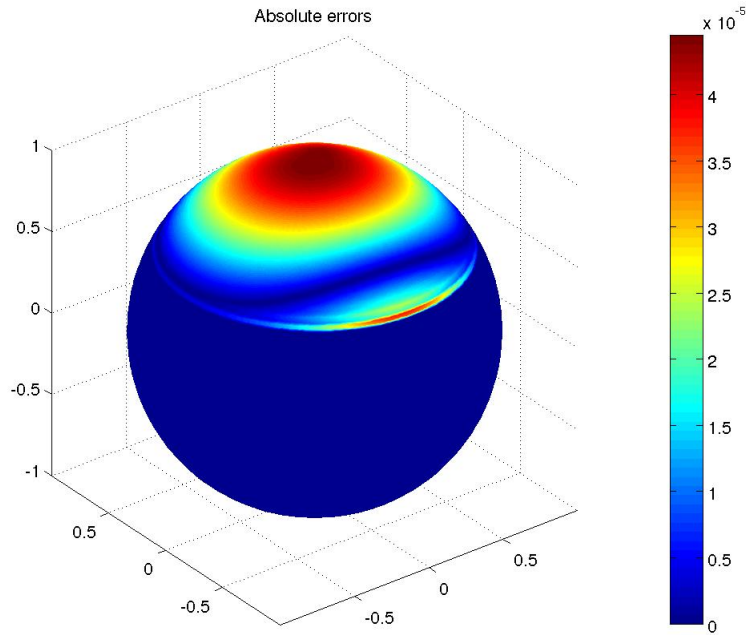


FIG. 4.3. Absolute errors with $M = 4000$ and $N = 4200$

- [12] W. Freeden, T. Gervens, M. Schreiner: Constructive Approximation on the Sphere (with Applications to Geomathematics), Clarendon Press, Oxford, 1998.
- [13] S. Gemmrich, N. Nigam, O. Steinbach, Boundary integral equations for the Laplace-Beltrami operator, in Mathematics and Computation, a Contemporary View, Proceedings of the Abel Symposium 2006, Vol. 3 (Eds: H. Munthe-Kaas and B. Owren), 21–37, Springer, Heidelberg.
- [14] A. E. Gill, Atmosphere-Ocean Dynamics, International Geophysics Series Volume 30, Academic, New York, 1982.
- [15] T. Hangelbroek, F. J. Narcowich, J. D. Ward, Polyharmonic and related kernels on manifolds: interpolation and approximation, Found. Comput. Math., **12** (2012), 625–670.
- [16] E. Hebey, Nonlinear analysis on manifolds: Sobolev spaces and inequalities, Courant Lecture Notes in Mathematics, Amer. Math. Soc. 2000.
- [17] Y. C. Hon and X. Z. Mao, An efficient numerical scheme for Burgers' equation. Appl. Math. Comput., **95** (1998), 37–50.
- [18] Y. C. Hon and R. Schaback, On unsymmetric collocation by radial basis functions. Appl. Math. Comput., **119** (2001), 177–186.
- [19] S. Hubbert and T. M. Morton. A Duchon framework for the sphere. *J. Approx. Theory*, 129:28–57, 2004.
- [20] E. J. Kansa, Multiquadrics - A scattered data approximation scheme with applications to computational fluid-dynamics i. Comput. Math., **19** (1990) 127–145.
- [21] E. J. Kansa, Multiquadrics - A scattered data approximation scheme with applications to computational fluid-dynamics ii: solutions to parabolic, hyperbolic and elliptic partial differential equations. Comput. Math., **19** (1990) 147–161.
- [22] Q. T. Le Gia. Galerkin approximation for elliptic PDEs on spheres. *J. Approx. Theory*, 130:123–147, 2004.
- [23] Q. T. Le Gia, F. J. Narcowich, J. D. Ward, H. Wendland, Continuous and discrete least-squares approximation by radial basis functions on spheres, *J. Approx. Theory*, **143** (2006), 124–133.
- [24] J. L. Lions, E. Magenes, Non-homogeneous boundary value problems and applications, Vol. I, Springer-Verlag, New York, 1972.
- [25] M. Mitrea, M. Taylor, Boundary layer methods for Lipschitz domains in Riemannian manifolds, *J. Funct. Anal.*, **163** (1999), 181–251.
- [26] T. M. Morton, M. Neamtu, Error bounds for solving pseudodifferential equations on spheres by

- collocation with zonal kernels, *J. Approx. Theory*, **114** (2002), 242–268.
- [27] C. Müller, *Spherical harmonics*, Lecture Notes in Mathematics, Vol. 17, New York, Springer-Verlag (1966).
 - [28] F. J. Narcowich, J. D. Ward, Scattered data interpolation on spheres: error estimates and locally supported basis functions, *SIAM J. Math. Anal.*, **33**(6) (2002), 1393–1410.
 - [29] R. Kidambi, P. K. Newton, Point vortex motion on a sphere with solid boundaries, *Physics of Fluids*, **12**, no. 3 (2000).
 - [30] P. Pucci, J. Serrin, Review: The strong maximum principle revisited, *J. Differential Equations*, **196** (2004), 1–66.
 - [31] E. B. Saff, E. A. Rakhmanov, and Y. M. Zhou. Minimal discrete energy on the sphere, *Mathematical Research Letters*, **1** (1994), 647–662.
 - [32] L. Saloff-Coste, Pseudo-Poincaré inequalities and applications to Sobolev inequalities, *Around the research of Vladimir Maz'ya I*, Function Spaces, (Ed.) A. Laptev, 349–359.
 - [33] I. J. Schoenberg, Positive definite function on spheres, *Duke Math. J.*, **9** (1942), 96–108.
 - [34] J. G. Ratcliffe. *Foundations of Hyperbolic Manifolds*. Springer, New York, 1994.
 - [35] H. Wendland, Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree, *Adv. in Comp. Math.*, **4** (1995), 389–396.
 - [36] H. Wendland, *Scattered Data Approximation*, Cambridge University Press, Cambridge, 2005.
 - [37] J. Wloka, *Partial differential equations*, Cambridge University Press, Cambridge, UK, 2005.
 - [38] Y. Xu, E. W. Cheney, Strictly positive definite functions on spheres, *Proc. Amer. Math. Soc.*, **116** (1992), 977–981.